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Extended chiral algebras in the $\mathrm{SU}(2)_0$ WZNW model

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Abstract: We investigate the W-algebras generated by the integer dimension chiral primary operators of the $\mathrm{SU}(2)_0$ WZNW model. These have a form almost identical to that found in the $c = -2$ model but have, in addition, an extended affine Lie algebraic structure. Moreover on hamiltonian reduction these $\mathrm{SU}(2)_0$ W-algebras exactly reduce to those found in $c = -2$. We explicitly find the free field representations for the chiral $j = 2$ and $j = 3$ operators which have, respectively, a fermionic doublet and bosonic triplet nature. The correlation functions of these operators accounts for the rational solutions of the Knizhnik-Zamolodchikov equation that we find. We explicitly compute the full algebra of the $j = 2$ operators and find that the associativity of the algebra is only guaranteed if certain null vectors decouple from the theory. We conjecture that these algebras may produce a quasi-rational conformal field theory.

Keywords: Field Theories in Lower Dimensions, Conformal and W Symmetry, Integrable Field Theories.

Contents

1. Introduction

The study of conformal invariance in two dimensions has been a fascinating and productive area of research for the last twenty years [1]. There is an interesting class of conformal field theories (CFTs) called logarithmic conformal field theories (LCFTs). In these theories the irreducible primary operators do not close under fusion and indecomposable representations are inevitably generated [2]. The operators in the theory have scaling dimensions that are either degenerate or differ by integers. In these cases it is possible to have a non-trivial Jordan block structure.

LCFTs have emerged in many different areas for example: WZNW models and gravitational dressing [3]–[10], polymers [11, 12, 13], disordered systems and the Quantum Hall effect [14]–[29], string theory [30]–[39], 2d turbulence [40]–[44], multi-colour QCD at lowx [45] and the Seiberg-Witten solution of $\mathcal{N} = 2$ SUSY Yang-Mills [46, 47]. Deformed LCFTs, Renormalisation group flows and the c-theorem were discussed in [14, 48, 49]. The holographic relation between logarithmic operators and vacuum instability was considered in [50, 51, 52].

There has also been much work on analysing the general structure and consistency of such models in particular the $c_{p,q}$ models and the special case of $c = -2$ which is by far the best understood [53]–[57]. It is unclear as yet how much of the structure, for instance the role of extended algebras, is generic to all LCFTs. For more about the general structure of LCFT see [58]–[61] and references therein. Introductory lecture notes on LCFT and more references can be found in [62]–[65]. The WZNW model is of great importance in CFT. Correlation functions in such models obey differential, Knizhnik-Zamolodchikov (KZ), equations [66] coming from null states in the theory. The solutions to the KZ equations and correlation functions for the integrable sector of the $\mathrm{SU}(2)_k$ model were studied by [67, 68]. There is a simple Dotsenko-Fateev integral representation for solutions but these do not converge in many cases beyond the integrable representations. In particular in the cases in which logarithms appear we have to be very careful when analytically continuing the solutions.

The $\mathrm{SU}(2)_0$ WZNW model was studied in [30] in the context of describing NS-5 branes. Later $\mathrm{SU}(2)_0$, and its supersymmetric extension, were studied in more detail [4]. It was found that there were logarithmic terms in the four point correlation function of the fundamental spin 1/2 primaries. This model was re-examined recently [9, 10] and we shall follow up on a few points which were studied there: the algebra of the integer dimension fields, and the hamiltonian reduction procedure. One of the interesting features of LCFT is the existence of a hidden symmetry which means there are extra fields with integer conformal dimensions [14]. In the case of the integrable representations these were previously studied in [69, 70] in which it was found that the rational solutions to the KZ equation were in one-one correspondence with the extensions of the chiral algebra. In the $\mathrm{SU}(2)_0$ model the integrable sector is trivial (the affine current J^a is a null vector) and one is forced to immediately consider non-integrable representations.

In this article we wish to demonstrate that the $\mathrm{SU}(2)_0$ WZNW model possesses several chiral algebras, beyond the normal affine Lie algebra, that are generated by the integer dimension fields. We shall construct the lowest lying currents directly in the the free field approach and shall discover that the fields naturally become extended multiplets in exact analogue to the $c = -2$ model. We shall then demonstrate how such an extra extended multiplet structure emerges from the rational solutions of the KZ equation (The $j = 1$ solutions are a special case of a more general solution recently found in [71]). We shall finally show that upon hamiltonian reduction these multiplets become precisely those previously found in the $c = -2$ triplet model. This strongly suggests that such currents could be used to construct a quasi-rational $\mathrm{SU}(2)_0$ theory in exact analogue to the triplet model of $c = -2$ [55]. We shall briefly comment on the expected structure of the general extended $\mathrm{SU}(2)_k$ models for $k \in N$. In appendix A we have explicitly given the solutions for the rational correlators with $j_i = 3$. We have also given the (rather lengthy!) OPEs for the $j = 2$ fields found using our free field expressions.

2. Extended chiral algebras

2.1 Free field representation

We shall use the standard Wakimoto representation [72] for the affine Lie algebra $\widehat{\mathrm{SU}(2)}$ specialised to the case $k = 0$. Our currents are:

$$J^+ = \beta\,, \qquad J^3 = i\partial\phi + \gamma\beta\,, \qquad J^- = -2i\partial\phi\gamma - \beta\gamma^2\,, \tag{2.1}$$

where β and γ obey the standard free field relations:

$$\beta(z)\beta(w) \sim 0 \sim \gamma(z)\gamma(w)\,, \qquad \beta(z)\gamma(w) \sim \frac{-1}{z-w} \sim -\gamma(z)\beta(w) \tag{2.2}$$

and ϕ is a free boson:

$$\phi(z)\phi(w) \sim -\ln(z-w) \tag{2.3}$$

these obey the standard affine Lie algebra with zero central extension:

$$J^3(z)J^\pm(w) \sim \pm\frac{J^\pm(w)}{z-w}\,, \qquad J^+(z)J^-(w) \sim \frac{2J^3(w)}{z-w}\,, \qquad J^3(z)J^3(w) \sim 0\,. \tag{2.4}$$

The stress tensor is just the standard Sugawara one:

$$T = \frac{1}{2}\left(\frac{1}{2}J^+J^- + \frac{1}{2}J^-J^+ + J^3J^3\right) = -\beta\partial\gamma - \frac{1}{2}\partial\phi\partial\phi + \frac{i}{2}\partial^2\phi\,. \tag{2.5}$$

We now observe that the ϕ part of this is none other than the $c = -2$ system. This fact will be essential in order to see the extended multiplet nature in the free field representations. We can thus write:

$$\xi \sim e^{i\phi}\,, \qquad \eta \sim e^{-i\phi}\,, \tag{2.6}$$

where ξ, η is the symplectic fermion system:

$$\xi(z)\eta(w) \sim \frac{1}{z-w} \sim \eta(z)\xi(w)\,, \qquad i\partial\phi = \xi\eta\,. \tag{2.7}$$

The currents and stress tensor become:

$$\begin{aligned} J^+ &= \beta\,, \qquad J^3 = \xi\eta + \beta\gamma\,, \qquad J^- = -2\xi\eta\gamma - \beta\gamma^2\,, \\ T &= -\beta\partial\gamma - \xi\partial\eta\,. \end{aligned} \tag{2.8}$$

The affine Lie algebra primary fields Φ_j obey:

$$J^a(z)\Phi_j(w) \sim \frac{t^a\Phi_j(w)}{z-w}\,, \tag{2.9}$$

where t^a is a matrix in the spin j representation of $\mathrm{SU}(2)$. They have conformal dimension:

$$h = \frac{j(j+1)}{2}\,. \tag{2.10}$$

Normally the vertex operators of primary affine Lie algebra fields are given by:

$$\Phi_{j,m} = (-\gamma)^{j-m}e^{ij\phi}\,. \tag{2.11}$$

However, in order to calculate correlators with such operators, vacuum charges and screening operators must be inserted [73, 74]. This is in general quite complicated moreover the result is in general only an integral representation. As we shall see for certain operators in $\mathrm{SU}(2)_0$ one can obtain correlators and OPEs in a much simpler way.

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2.2 Extended algebras

For $j \in N$ we have $h \in N$ and so one may suspect that there is a simpler way of evaluating correlators for such fields. We found the following expressions for the fields.[1] For brevity we shall only write the highest component, that is the one annihilated by J^+. All others in the multiplet may be obtain by the action of J^- (for $j = 2$ we list the full multiplets in appendix A):

- $j = 1$

$$J^+ = \beta \,, \tag{2.12}$$

- $j = 2$

$$\begin{aligned} G^{++} &= -\beta\partial\xi + \partial\beta\xi \,, \\ H^{++} &= \beta\beta\beta\eta \,, \end{aligned} \tag{2.13}$$

- $j = 3$

$$\begin{aligned} X^{+++} &= \beta\beta\beta\beta\beta\partial\eta\eta \,, \\ Y^{+++} &= \frac{1}{2}\beta\beta\beta\eta\partial^2\xi - \frac{1}{2}\beta\beta\beta\partial\eta\partial\xi - \frac{3}{2}\partial\beta\beta\beta\eta\partial\xi + \frac{1}{2}\partial\beta\beta\beta\partial\eta\xi + \frac{3}{2}\partial\beta\partial\beta\beta\eta\xi + \\ &\quad + \partial\beta\partial\beta\partial\beta - \frac{1}{2}\partial^2\beta\beta\beta\eta\xi - \frac{3}{4}\partial^2\beta\partial\beta\beta + \frac{1}{12}\partial^3\beta\beta\beta \,, \\ Z^{+++} &= -\beta\partial^2\xi\partial\xi + \partial\beta\partial^2\xi\xi - \partial^2\beta\partial\xi\xi \,. \end{aligned} \tag{2.14}$$

It is a non-trivial fact that these actually belong to the *discrete* representations of SU(2) and thus the lowest component in the multiplet is annihilated by J^-. We have also included the $j = 1$ field (2.12) which is of course just the affine current J^+ itself. We also found other solutions $\xi, \partial\xi\xi, \partial^2\xi\partial\xi\xi$ for $j = 1, 2, 3$, respectively. They are in fact extra primary operators that have different behaviour under crossing symmetries. For instance the extra $j = 1$ field is one of the fermionic operators found in [9]. We shall not discuss these operators here as these are not responsible for the rational solutions to the KZ equation that we shall discuss later. To summarize: the use of the symplectic fermion system (2.8) produces exactly the correct chiral field content of the $SU(2)_0$ model (We have checked this explicitly up $h = 6$ operators). We shall find that these chiral fields are also in one-one correspondence with the rational solutions of the KZ equation.

2.3 Operator product expansions

We can immediately observe that $j = 1, 2, 3$ are alternately bosonic, fermionic, bosonic and that there are, respectively, a singlet, doublet and a triplet of possible fields. We remind the reader once again that each of the two solutions for $j = 2$ yields a full SU(2) multiplet with $m = +2, \ldots, m = -2$ and that we have only written the $m = +2$ terms. Although the full OPEs are quite complicated (see appendix A for full $j = 2$ algebra) we see that by

[1] We are extremely grateful to K. Thielemans for giving us a copy of his OPEDEFS Mathematica program [75] which was invaluable in finding these expressions.

using the above symmetries and the fact that a chiral algebra is completely determined by its the singular terms we can already deduce almost all the structure. We know:

$$[j=0] \times [j=J] = [j=J]\,, \tag{2.15}$$

where [ ] denotes operators and all their affine Lie algebra descendents (In particular the $j=1$ state is of course just a descendent of the identity). A special case of this is (2.9). Now we shall find it convenient to use indices α, β for extra doublet index structure and a, b, c for the triplet. We now want to analyse the OPE of $[j=2]^\alpha$ with itself. Clearly as these are both fermionic the result must be bosonic. Furthermore the $j=3$ operators have $h=6$ and so cannot occur in the singular terms. Thus we must have:

$$[j=2]^\alpha \times [j=2]^\beta = d^{\alpha\beta}[j=0]\,, \tag{2.16}$$

where $d^{\alpha\beta}$ is the anti-symmetric tensor $d^{+-} = -d^{-+} = 1$.We have explicitly calculated the entire algebra of the $[j=2]^\pm$ operators (see appendix A) and have verified that it is indeed of the above form.

By similar arguments we find:

$$[j=2]^\alpha \times [j=3]^a = t^{a\alpha}_\beta [j=2]^\beta\,, \tag{2.17}$$

where $t^{a\alpha}_\beta$ are 2×2 matrices which generate the SU(2) Lie algebra. In this case we can only produce fermionic operators in the singular terms with dimension at most $3+6=9$. In particular this excludes the fermionic $j=4$ operator with $h=10$. Similarly:

$$[j=3]^a \times [j=3]^b = g^{ab}[j=0] + f^{ab}_c [j=3]^c\,, \tag{2.18}$$

where now g^{ab} and f^{ab}_c are, respectively, the metric and structure constants of SU(2). We thus expect $j=1,2,3$ to yield a closed W-algebra. We also expect that in the absence of other symmetries, such as orbifolding, we cannot get a finite W-algebra involving $j>3$. Unfortunately using the free field representations it became prohibitively difficult to go beyond the $j=2$ algebra.

3. Knizhnik-Zamolodchikov equation

3.1 Auxiliary variables

It will be convenient in much of this paper to introduce the following representation for the SU(2) generators [67]:

$$J^+ = x^2\frac{\partial}{\partial x} - 2jx\,, \qquad J^- = -\frac{\partial}{\partial x}\,, \qquad J^3 = x\frac{\partial}{\partial x} - j\,. \tag{3.1}$$

There is also a similar algebra in terms of $\bar{x}$ for the antiholomorphic part. It is easily verified that these obey the global SU(2) algebra.

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We introduce primary fields, $\phi_j(x,z)$ of the affine Lie algebra. Then:

$$\begin{aligned} J^+(z)\phi_j(x,w) &\sim \frac{\left(x^2\frac{\partial}{\partial x}-2jx\right)\phi_j(x,w)}{z-w}\,, \\ J^-(z)\phi_j(x,w) &\sim \frac{-\frac{\partial}{\partial x}\phi_j(x,w)}{z-w}\,, \\ J^3(z)\phi_j(x,w) &\sim \frac{\left(x\frac{\partial}{\partial x}-j\right)\phi_j(x,w)}{z-w}\,. \end{aligned} \tag{3.2}$$

The fields $\phi_j(x,z)$ are also primary with respect to the Virasoro algebra with L_0 eigenvalue:

$$h=\frac{j(j+1)}{k+2}\,. \tag{3.3}$$

where k is the level of the SU(2) WZNW model.

The two point functions are fully determined using global SU(2) and conformal transformations.

$$\langle\phi_{j_1}(x_1,z_1)\phi_{j_2}(x_2,z_2)\rangle = A(j_1)\delta_{j_1j_2}x_{12}^{2j_1}z_{12}^{-2h}\,. \tag{3.4}$$

We do not set $A(j)=1$ as there will be some zero-norm fields that do not decouple. The general form of the three point function is:

$$\begin{aligned} \langle\phi_{j_1}(x_1,z_1)\phi_{j_2}(x_2,z_2)\phi_{j_3}(x_3,z_3)\rangle = {} & C(j_1,j_2,j_3)\; x_{12}^{j_1+j_2-j_3}x_{13}^{j_1+j_3-j_2}x_{23}^{j_2+j_3-j_1}\times \\ & \times z_{12}^{-h_1-h_2+h_3}z_{13}^{-h_1-h_3+h_2}z_{23}^{-h_2-h_3+h_1}\,. \end{aligned} \tag{3.5}$$

The $C(j_1,j_2,j_3)$ are the structure constants which in principle completely determine the entire theory.

For the case of the four point correlation functions of SU(2) primaries the form is determined by global conformal and SU(2) transformations up to a function of the cross ratios. Our convention is:

$$\begin{aligned} \langle\phi_{j_1}(x_1,z_1)\phi_{j_2}(x_2,z_2)\phi_{j_3}(x_3,z_3)\phi_{j_4}(x_4,z_4)\rangle = {} & z_{43}^{h_2+h_1-h_4-h_3}z_{42}^{-2h_2}z_{41}^{h_3+h_2-h_4-h_1}\times \\ & \times z_{31}^{h_4-h_1-h_2-h_3}x_{43}^{-j_2-j_1+j_4+j_3}x_{42}^{2j_2}\times \\ & \times x_{41}^{-j_3-j_2+j_4+j_1}x_{31}^{-j_4+j_1+j_2+j_3}F(x,z)\,. \end{aligned} \tag{3.6}$$

Here the invariant cross ratios are:

$$x=\frac{x_{21}x_{43}}{x_{31}x_{42}}\,,\qquad z=\frac{z_{21}z_{43}}{z_{31}z_{42}}\,. \tag{3.7}$$

3.2 The Knizhnik-Zamolodchikov equation

Correlation functions of the WZNW model satisfy a set of partial differential equations known as Knizhnik-Zamolodchikov (KZ) equation due to constraints from the null states following from (2.5). These are:

$$|\chi\rangle=\left(L_{-1}-\frac{1}{k+2}\eta_{ab}J^a_{-1}J^b_0\right)|\phi\rangle\,. \tag{3.8}$$

For two and three point functions this gives no new information. However for the four point function (3.6) it becomes a partial differential equation for $F(x,z)$. For a compact Lie group this equation can be solved [66] as it reduces to a set of ordinary differential equations.

$$\left[(k+2)\frac{\partial}{\partial z_i}+\sum_{j\neq i}\frac{\eta_{ab}t_i^a\otimes t_j^b}{z_i-z_j}\right]\langle\phi_{j_1}(z_1)\cdots\phi_{j_n}(z_n)\rangle=0\,. \tag{3.9}$$

If we now use our representation (3.1) we find the KZ equation for four point functions is:

$$(k+2)\frac{\partial}{\partial z}F(x,z)=\left[\frac{\mathcal{P}}{z}+\frac{\mathcal{Q}}{z-1}\right]F(x,z)\,. \tag{3.10}$$

Explicitly these are:

$$\begin{aligned}
\mathcal{P} &= -x^2(1-x)\frac{\partial^2}{\partial x^2}+((-j_1-j_2-j_3+j_4+1)x^2+2j_1x+2j_2x(1-x))\frac{\partial}{\partial x}+\\
&\quad+2j_2(j_1+j_2+j_3-j_4)x-2j_1j_2\,,\\
\mathcal{Q} &= -(1-x)^2x\frac{\partial^2}{\partial x^2}-((-j_1-j_2-j_3+j_4+1)(1-x)^2+2j_3(1-x)+2j_2x(1-x))\frac{\partial}{\partial x}+\\
&\quad+2j_2(j_1+j_2+j_3-j_4)(1-x)-2j_2j_3\,.
\end{aligned} \tag{3.11}$$

4. Rational solutions to the $\mathrm{SU}(2)_0$ KZ equation

For the case of $j\in\mathbb{N}$ we found for all cases studied that we have either rational or logarithmic solutions. As we are discussing chiral algebras we shall concentrate on the rational solutions.

For a four point correlator involving a discrete representation of spin j we can write the general solution to the KZ equation as:

$$F(x,z)=F_0(z)+xF_1(z)+x^2F_2(z)+\cdots+x^{2j}F_{2j}(z)\,. \tag{4.1}$$

This allows one to reduce the KZ equation to a linear *ordinary* differential equation of order $2j+1$.

In [10] an ansatz was found for the four point correlator with $j_1=j_2=j_3=j_4=j$. The lowest component was:

$$F_0(z)=z^{-j(j-1)}(1-z)^{-j(j+1)}{}_2F_1(j,-j;1;z)\,. \tag{4.2}$$

Given this all other components are fully determined by substitution into the KZ equation. For $j\in\mathbb{N}$ these become rational solutions and are thus everywhere single-valued (i.e. they have trivial monodromy properties). Now using the crossing symmetries one can easily find a set of rational solutions on which one can perform the conformal bootstrap.

4.1 Spin $j=1$

For $j_1=j_2=j_3=j_4=1$ we find only one rational solution:

$$F_{1111}(x,z)=-\frac{1}{2(z-1)}+\frac{x}{z}+\frac{x^2}{2z(z-1)}\,. \tag{4.3}$$

This obeys:

$$\begin{aligned} F_{1111}(1-x,1-z) &= F_{1111}(x,z)\,, \\ z^{-2h}x^{2j}F_{1111}\left(\frac{1}{x},\frac{1}{z}\right) &= F_{1111}(x,z)\,. \end{aligned} \tag{4.4}$$

As this chiral solution is already single valued it serves as a perfectly good solution for the correlation function of four identical $j=1$, $\bar{j}=0$ fields. Substitution of the solution (4.3) into (3.6) and reverting to the index notation [9] gives us the correlator:

$$\langle \phi^a(z_1)\phi^b(z_2)\phi^c(z_3)\phi^d(z_4)\rangle = \left[\frac{\delta^{ab}\delta^{cd}}{z_{13}z_{14}z_{23}z_{24}} + \frac{\delta^{ac}\delta^{bd}}{z_{12}z_{34}z_{14}z_{32}} + \frac{\delta^{ad}\delta^{bc}}{z_{13}z_{12}z_{43}z_{42}}\right]. \tag{4.5}$$

If we wish to identify each of these fields this with the affine current $J^a(z)$ field (which of course has $j=1$, $\bar{j}=0$) then we have a problem as from the affine Lie algebra (2.4) we get:

$$\left\langle J^a(z_1)J^b(z_2)J^c(z_3)J^d(z_4)\right\rangle = 0\,. \tag{4.6}$$

The resolution is that there should be some $(h=1,\bar{h}=0)$ field N^a (such a field was found in [4]) which is a logarithmic partner to J^a. This obeys:

$$J^a(z)N^b(w,\bar{w}) \sim \frac{\delta^{ab}}{(z-w)^2} + \frac{i\epsilon^{abc}J^c}{z-w} + \cdots\,. \tag{4.7}$$

As we shall only have to insert one N^a field into these OPEs we can treat it as a chiral field even if it is not (A similar phenomena occurs with a $(2,0)$ logarithmic partner to T in $c=0$ theories [76]). Using (4.7) and the affine Lie algebra we can easily deduce:

$$\langle J^a(z_1)J^b(z_2)J^c(z_3)N^d(z_4)\rangle = \left[\frac{\delta^{ab}\delta^{cd}}{z_{13}z_{14}z_{23}z_{24}} + \frac{\delta^{ac}\delta^{bd}}{z_{12}z_{34}z_{14}z_{32}} + \frac{\delta^{ad}\delta^{bc}}{z_{13}z_{12}z_{43}z_{42}}\right]. \tag{4.8}$$

We still have full crossing symmetry and expanding this reveals the correct JJ OPEs. In fact we need not explicitly discuss N^a other than assume that such a field exists which makes the four point functions non zero so that we can use the correlators to obtain the OPEs.

4.2 Spin $j=2$

For the four point correlator with $j_1=j_2=j_3=j_4=2$ we find two rational solutions:

$$\begin{aligned} F^{(1)}_{2222}(x,z) &= \frac{1}{z^5}\big((-z^3-2z^4)+(-12z^2-16z^3+4z^4)x+(-18z+18z^3)x^2+ \\ &\qquad +(-4+16z+12z^2)x^3+(2+z)x^4\big)\,, \\ F^{(2)}_{2222}(x,z) &= \frac{1}{(z-1)^5}\big((10z-15z^2+9z^3-2z^4)+(60-140z+108z^2-36z^3+4z^4)x+ \\ &\qquad +(-90+162z-90z^2+18z^3)x^2+(36-44z+12z^2)x^3+ \\ &\qquad +(-3+z)x^4\big)\,. \end{aligned} \tag{4.9}$$

These obey:

$$
\begin{aligned}
F^{(1)}_{2222}(1-x,1-z) &= F^{(2)}_{2222}(x,z)\,, \\
F^{(2)}_{2222}(1-x,1-z) &= F^{(1)}_{2222}(x,z)\,, \\
x^4 z^{-6} F^{(1)}_{2222}\left(\frac{1}{x},\frac{1}{z}\right) &= -F^{(1)}_{2222}(x,z)\,, \\
x^4 z^{-6} F^{(2)}_{2222}\left(\frac{1}{x},\frac{1}{z}\right) &= -F^{(1)}_{2222}(x,z) + F^{(2)}_{2222}(x,z)\,.
\end{aligned} \tag{4.10}
$$

We see that from the leading forms of these solutions that they do not go as z^{-6} as one would expect from an $h = 3$ field. Therefore as there is no contribution from the identity field in the OPE the two point functions of these fields must vanish. This can also be verified from the explicit free field expressions that we gave earlier (2.13). It is clear that a similar approach to the $j = 1$ fields must be carried out here as well but we shall not discuss this in detail here. From the conformal blocks (4.10) we can already see the fermionic nature of these fields as we expected.

For $j = 3$ we have three rational solutions (see appendix A). They exhibit a triplet nature as we expected. All these correlators agree with the leading forms of the OPEs from the free field representations (the subleading terms are technically much more complicated to verify) and again reveal that the two point function vanishes.

We can of course consider correlators where not all four fields have the same spin. For instance the correlator with $j_1 = 1$, $j_2 = 1$, $j_3 = 1$, $j_4 = 2$ yields two rational solutions:

$$
F^{(1)}_{1112}(x,z) = 1 - z + \frac{1-z}{z}x\,, \qquad F^{(2)}_{1112}(x,z) = \frac{1}{1-z} + \frac{1-2z}{z(1-z)}x\,. \tag{4.11}
$$

However neither of these leads to a chiral correlator respecting the full crossing symmetries if the $j = 1$ operators are all identical. This was to be expected as the fermionic nature of the chiral $j = 2$ operator means this correlator must vanish.

5. Hamiltonian reduction

There is another interesting way in which $\mathrm{SU}(2)_0$ is related to the $c = -2$ theory. When we do a quantum hamiltonian reduction of $\mathrm{SU}(2)_k$ theories normally by imposing the constraint $J^+ \sim 1$ it is well known [78]–[82] that we get the $c_{k+2,1}$ minimal models.

The central charge and conformal weights of the reduced theory are given by:

$$
c = c_{k+2,1} = 13 - 6\left(k + 2 + \frac{1}{k+2}\right), \tag{5.1}
$$

$$
h = \frac{j(j+1)}{k+2} - j\,. \tag{5.2}
$$

Therefore $k = 0$ reduces to $c_{2,1} = -2$.

Here we follow an elegant realisation of this reduction that allow us to perform this at the level of the correlation functions. The motivation for such a procedure comes from the observation that setting $x_i = z_i$ in the two and three point functions (3.4) and (3.5) gives the expected results with the correct conformal weight (5.2).

What is far from obvious and is the central result of a series of papers [83, 84, 85, 86] is that this simple procedure also extends to the four point functions. It was shown that, if instead of expanding as in (4.1), we expand about the point $x = z$:

$$F(x,z) = \tilde{F}_0(z) + (x-z)\tilde{F}_1(z) + (x-z)^2\tilde{F}_2(z) + \cdots + (x-z)^{2j}\tilde{F}_{2j}(z) \tag{5.3}$$

then the lowest component $\tilde{F}_0(z)$ obeys the correct differential equation for the field $h_{1,2j+1} = \frac{j(j+1)}{k+2} - j$. This can also obtained from the solution $F(x,z)$ by taking $x_i = z_i$ which gives $x = z$. As this is at the level of conformal blocks overall normalisations are not important, at least for individual blocks, although these can become singular in some cases [77].

Several examples of this were demonstrated in [10] for the half-integer spin cases involving logarithms in the $\mathrm{SU}(2)_0$ WZNW model. Here we shall instead discuss the chiral solutions of the integer spin operators. For the case of $k = 0$ we get:

$$\begin{array}{ccccc} j & 0 & 1 & 2 & 3, \\ h & 0 & 0 & 1 & 3. \end{array} \tag{5.4}$$

Now performing such a reduction on the F_{1111} correlator (4.3) we find (up to a numerical factor):

$$F_{1111}(x,z) \longrightarrow 1\,. \tag{5.5}$$

Such a simple correlator is due to the fact that the $j = 1$ field reduces to an $h = 0$ field. This does not generate a chiral algebra in the reduced theory as it is just the identity field Ω. In fact in the $c = -2$ theory the identity is a zero-norm field [14]:

$$\langle \Omega(z_1)\Omega(z_2)\rangle = 0 \tag{5.6}$$

and so in order to reproduce the conformal block one must insert the logarithmic partner of the identity ω:

$$\langle \Omega(z_1)\omega(z_2,\bar{z}_2)\rangle = 1\,. \tag{5.7}$$

We then have:

$$\langle \Omega(z_1)\Omega(z_2)\Omega(z_3)\omega(z_4,\bar{z}_4)\rangle = 1\,. \tag{5.8}$$

We see that the discussion of insertion of logarithmic partners to get the conformal blocks in $\mathrm{SU}(2)_0$ parallels that of $c = -2$. However there is an important difference: in $c = -2$ this indecomposable representation is at the $h = 0$ level only whereas for $\mathrm{SU}(2)_0$ it is believed that the indecomposable representations occur between the $j = 0$ and $j = 1$ states. From (5.4) we see that both $j = 0, 1$ fields reduce to $h = 0$ fields.

For the F_{2222} correlators (4.9) we get:

$$F^{(1)}_{2222}(x,z) \longrightarrow 1 - \frac{1}{z^2}\,, \qquad F^{(2)}_{2222}(x,z) \longrightarrow 1 - \frac{1}{(1-z)^2}\,. \tag{5.9}$$

These reproduce precisely the conformal blocks of the $h_{1,2} = 1$ operators in the $c = -2$ theory as claimed. The doublet nature is unaffected by the reduction. Thus the hamiltonian reduction of the multiplets G, H (2.13) gives precisely the fields $\chi^\pm$ which are the chiral fermionic doublet in $c = -2$:

$$\chi\pm(z)\chi\pm(w) \sim 0\,, \qquad \chi^{+}(z)\chi^{-}(w) \sim \frac{1}{(z-w)^2}\,. \tag{5.10}$$

The symplectic fermion algebra is automatically associative in other words the Jacobi identity is automatically satisfied given the chiral algebra. However for the $j = 2$ algebra in $\mathrm{SU}(2)_0$ we found that it was only associative **provided** that certain null vectors decouple. For example:

$$\mathcal{N} = 4J^3G^{++} + J^{+}G^{+} - 2\partial G^{++}\,. \tag{5.11}$$

It can be verified that this decouples, as it must do, using the free field representation of the operators given earlier. However from the algebraic point of view the algebra is only consistent if such null vectors decouple. We hope this may give some insight into how to form a quasi-rational $\mathrm{SU}(2)_0$ theory.

Reduction of the F_{3333} correlators (A.3) gives:

$$\begin{aligned}
F^{(1)}_{3333} &\longrightarrow \frac{1}{(z-1)^6} z^4\left(6 - 6z + z^2\right),\\
F^{(2)}_{3333} &\longrightarrow \frac{1}{z^6(z-1)^6}\left(2 - 12z + 12z^2 + 50z^3 - 225z^4 + 468z^5 - 588z^6 + 468z^7 - 225z^8 + \right.\\
&\qquad\qquad \left. + 50z^9 + 12z^{10} - 12z^{11} + 2z^{12}\right),\\
F^{(3)}_{3333} &\longrightarrow \frac{1}{z^6}\left(1 - 9z^2 + 16z^3 - 9z^4 + z^6\right).
\end{aligned} \tag{5.12}$$

These again, as expected, reproduce the conformal blocks for $h_{1,3} = 3$ operators in $c = -2$. More explicitly by analysing the pole structure and symmetries we find that these are related (up to normalisation) in the following way to the standard fields [53]:

$$\begin{aligned}
&F^{(1)}_{3333} \longrightarrow \left\langle W^{+}(0)W^{+}(z)W^{-}(1)W^{-}(\infty)\right\rangle, \qquad F^{(2)}_{3333} \longrightarrow \left\langle W^{3}(0)W^{3}(z)W^{3}(1)W^{3}(\infty)\right\rangle,\\
&F^{(3)}_{3333} \longrightarrow \left\langle W^{+}(0)W^{-}(z)W^{-}(1)W^{+}(\infty)\right\rangle.
\end{aligned} \tag{5.13}$$

Again we are being slightly sloppy because, as we have discussed before, we must insert a logarithmic zero mode into these correlators to reproduce the conformal blocks. The first of these is the unique solution that has no singular terms as $z \to 0$. The third is similar but with no singular terms as $z \to 1$. The second is the unique solution invariant under all crossing symmetries. Analysing these is sufficient to fully reconstruct the algebra generated by the W^a fields [53]:

$$\begin{aligned}
T(z)T(w) &\sim \frac{-2}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,,\\
T(z)W^a(w) &\sim \frac{3W^a(w)}{(z-w)^2} + \frac{\partial W^a(w)}{z-w}\,,\\
W^a(z)W^b(w) &\sim g^{ab}\left(\frac{1}{(z-w)^6} - 3\frac{T(w)}{(z-w)^4} - \frac{3}{2}\frac{\partial T(w)}{(z-w)^3} + \frac{3}{2}\frac{\partial^2 T(w)}{(z-w)^2} - 4\frac{(T^2)(w)}{(z-w)^2} + \right.\\
&\qquad \left. + \frac{1}{6}\frac{\partial^3 T(w)}{z-w} - 4\frac{\partial(T^2)(w)}{z-w}\right) -\\
&\qquad - 5f^{ab}_{c}\left(\frac{W^c(w)}{(z-w)^3} + \frac{1}{2}\frac{\partial W^a(w)}{(z-w)^2} + \frac{1}{25}\frac{\partial^2 W^c(w)}{z-w} + \frac{1}{25}\frac{(TW^c)(w)}{z-w}\right).
\end{aligned} \tag{5.14}$$

The diagonal generator W^3 generates an automatically associative W-algebra that is precisely the Zamolodichikov $W(2,3)$ algebra [89] at $c = -2$. The full triplet algebra however is only associative if certain null vectors decouple [87]. We have shown that the above algebra **can** come from a hamiltonian reduction of an SU(2) structure. This is in complete contrast to the normal case of hamiltonian reduction of the SU(2) affine Lie algebra which leads to the Virasoro algebra only.

5.1 General $\mathrm{SU}(2)_k$ structure

We shall now comment briefly on the structure that we expect in $\mathrm{SU}(2)_k$ for $k \in \mathbb{N}$. We have in general an affine Lie algebra null vector that transforms in the $j = k+1$ representation. Imposing this as a null vector on states gives the normal rational affine Lie algebra theory with $0 \leq j \leq k/2$. For $\mathrm{SU}(2)_0$ this null vector was precisely the affine current J^a and this rational set was trivial. Using (5.2) we see that the reduction of this null vector gives an $h = 0$ state in the $c_{k+2,1}$ theory. We thus see that extending the model by adding a logarithmic partner which prevents the affine Lie algebra null vector decoupling automatically leads to a logarithmic partner for the vacuum in the $c_{p,1}$ models.

All the $c_{p,1}$ models have a triplet algebra generated by the $h_{1,4p-1} = h_{3,1} = 2p - 1$ fields [88]. These come from the hamiltonian reduction of $j = 2k+3$ fields in $\mathrm{SU}(2)_k$. We therefore conjecture that there will also be a triplet algebra generated by these fields and correspondingly, in general, three rational solutions to the KZ equation. However, as we previously mentioned, there remain many subtleties in the reduction process [77]. In the case of $c = -2$ there was also the $h_{2,1} = 1$ fields which generated a doublet algebra but this is not typical of the other $c_{p,1}$ models.

6. Conclusion

We have presented evidence, from both the free field construction and the explicit correlation functions, that there exist extended algebras in the $\mathrm{SU}(2)_0$ model. Their extra multiplet structure is identical to that found in $c = -2$. We have shown that expressing the $c = -2$ part of the $\mathrm{SU}(2)_0$ stress tensor in terms of symplectic fermions leads us to *exactly* the correct chiral operator content of the model. There is a chiral doublet of fermionic $j = 2$ fields and a triplet of bosonic $j = 3$ fields. Moreover hamiltonian reduction of these fields gives us precisely the $h = 1$ doublet and $h = 3$ triplet in $c = -2$. It is still an open question as to what extent the reduction extends to other operators of the theory.

In view of the fact that the triplet algebra in $c = -2$ yielded a quasi-rational model we may very much hope that these extended algebras will do the same for $\mathrm{SU}(2)_0$. We have verified that the $j = 2$ algebra is only consistent if certain null vectors decouple from the spectrum. It would be very interesting to be able to produce a full rational model of $\mathrm{SU}(2)_0$ as this may be useful in shedding light on the many questions that remain on the general structure at $c = 0$ [90].

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There are clearly many outstanding questions. It would be very interesting to investigate other $\widehat{\mathrm{SU}(2)}$ theories and in particular the case $k \in \mathbb{N}$ which we have already mentioned that we expect to have a similar structure to $c_{k+2,1}$. For $k \neq N$ the extended structure remains an interesting open problem.

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A. Appendix

A.1 Correlation functions

The correlation functions for $j_1 = j_2 = j_3 = j_4 = 3$ are:

$$
\begin{aligned}
F^{(1)}_{3333}(x,z) = {} & \frac{1}{462(z-1)^{11}} z^3 \times \\
& \times \big\{(-252z + 756z^2 - 910z^3 + 560z^4 - 190z^5 + 36z^6 - 3z^7) + \\
& \quad + (-2520 + 9576z - 14532z^2 + 11340z^3 - 4860z^4 + 1140z^5 - 138z^6 + 6z^7)x + \\
& \quad + (6300 - 21000z + 27510z^2 - 18000z^3 + 6150z^4 - 1050z^5 + 75z^6)x^2 + \\
& \quad + (-5600 + 16240z - 17840z^2 + 9200z^3 - 2200z^4 + 200z^5)x^3 + \\
& \quad + (2100 - 5010z + 4200z^2 - 1425z^3 + 150z^4)x^4 + \\
& \quad + (-300 + 516z - 258z^2 + 30z^3)x^5 + (10 - 8z + z^2)x^6\big\}\,, \qquad \text{(A.1)}
\end{aligned}
$$

$$
\begin{aligned}
F^{(2)}_{3333}(x,z) = {} & \frac{1}{231z^{11}(z-1)^{11}} \times \\
& \times \big\{(-z^5 + 8z^6 - 10z^7 - 70z^8 + 455z^9 - 1456z^{10} + 3003z^{11} - 4290z^{12} + 4290z^{13} - \\
& \quad - 2860z^{14} + 875z^{15} + 560z^{16} - 910z^{17} + 560z^{18} - 190z^{19} + 36z^{20} - 3z^{21}) + \\
& \quad + (-30z^4 + 258z^5 - 516z^6 - 1380z^7 + 11970z^8 - 40950z^9 + 88998z^{10} - \\
& \qquad - 134706z^{11} + 145860z^{12} - 111540z^{13} + 54510z^{14} - 7770z^{15} - 12180z^{16} + \\
& \qquad + 11340z^{17} - 4860z^{18} + 1140z^{19} - 138z^{20} + 6z^{21})x + \\
& \quad + (-150z^3 + 1425z^4 - 4200z^5 - 870z^6 + 46200z^7 - 184275z^8 + 436800z^9 - \\
& \qquad - 718575z^{10} + 858000z^{11} - 750750z^{12} + 471900z^{13} - 194775z^{14} + \\
& \qquad + 30450z^{15} + 21630z^{16} - 18000z^{17} + 6150z^{18} - 1050z^{19} + 75z^{20})x^2 + \\
& \quad + (-200z^2 + 2200z^3 - 9200z^4 + 13920z^5 + 26880z^6 - 197400z^7 + 564200z^8 - \\
& \qquad - 1058200z^9 + 1430000z^{10} - 1430000z^{11} + 1058200z^{12} - 564200z^{13} + \\
& \qquad + 197400z^{14} - 26880z^{15} - 13920z^{16} + 9200z^{17} - 2200z^{18} + 200z^{19})x^3 + \\
& \quad + (-75z + 1050z^2 - 6150z^3 + 18000z^4 - 21630z^5 - 30450z^6 + 194775z^7 - \\
& \qquad - 471900z^8 + 750750z^9 - 858000z^{10} + 718575z^{11} - 436800z^{12} + \\
& \qquad + 184275z^{13} - 46200z^{14} + 870z^{15} + 4200z^{16} - 1425z^{17} + 150z^{18})x^4 + \\
& \quad + (-6 + 138z - 1140z^2 + 4860z^3 - 11340z^4 + 12180z^5 + 7770z^6 - 54510z^7 +
\end{aligned}
$$

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$$
\begin{aligned}
&+ 111540z^8 - 145860z^9 + 134706z^{10} - 88998z^{11} + 40950z^{12} - 11970z^{13} + \\
&+ 1380z^{14} + 516z^{15} - 258z^{16} + 30z^{17})x^5 + \\
&+ (3 - 36z + 190z^2 - 560z^3 + 910z^4 - 560z^5 - 875z^6 + 2860z^7 - \\
&- 4290z^8 + 4290z^9 - 3003z^{10} + 1456z^{11} - 455z^{12} + 70z^{13} + \\
&+ 10z^{14} - 8z^{15} + z^{16})x^6\big\} ,
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
F^{(3)}_{3333}(x,z) &= \frac{1}{462z^{11}}(z-1)^3 \times \\
&\times \big\{(-z^5 - 6z^6 - 3z^7) + (-30z^4 - 162z^5 - 54z^6 + 6z^7)x + \\
&+ (-150z^3 - 675z^4 + 75z^6)x^2 + (-200z^2 - 600z^3 + 600z^4 + 200z^5)x^3 + \\
&+ (-75z + 675z^3 + 150z^4)x^4 + (-6 + 54z + 162z^2 + 30z^3)x^5 + \\
&+ (3 + 6z + z^2)x^6\big\} .
\end{aligned} \tag{A.3}
$$

A.2 Operator product expansions

Using the free field representation of the the affine Lie algebra generators and those of the $j = 2$ representations G and H we can explicitly find the full algebra. The top component G^{++} was given in the text (2.13). By operating repeatedly by J^- (in an obvious notation) one gets:

$$
\begin{aligned}
G^{++} &= -\beta\partial\xi + \partial\beta\xi , \qquad G^{+} = 4\beta\gamma\partial\xi + 4\eta\partial\xi\xi - 4\partial\beta\gamma\xi , \\
G^{0} &= -12\beta\gamma\gamma\partial\xi - 24\gamma\eta\partial\xi\xi + 12\partial\beta\gamma\gamma\xi , \\
G^{-} &= 24\beta\gamma\gamma\gamma\partial\xi + 72\gamma\gamma\eta\partial\xi\xi - 24\partial\beta\gamma\gamma\gamma\xi , \\
G^{--} &= -24\beta\gamma\gamma\gamma\gamma\partial\xi - 96\gamma\gamma\gamma\eta\partial\xi\xi + 24\partial\beta\gamma\gamma\gamma\gamma\xi .
\end{aligned} \tag{A.4}
$$

The H multiplet is given by:

$$
\begin{aligned}
H^{++} &= \beta\beta\beta\eta , \\
H^{+} &= -4\beta\beta\beta\gamma\eta + 6\beta\beta\partial\eta - 6\partial\beta\beta\eta , \\
H^{0} &= 12\beta\beta\beta\gamma\gamma\eta - 36\beta\beta\gamma\partial\eta + 36\beta\partial\eta\eta\xi + 6\beta\partial^2\eta + 36\partial\beta\beta\gamma\eta - 24\partial\beta\partial\eta + 6\partial^2\beta\eta , \\
H^{-} &= -24\beta\beta\beta\gamma\gamma\gamma\eta + 108\beta\beta\gamma\gamma\partial\eta - 216\beta\gamma\partial\eta\eta\xi - 36\beta\gamma\partial^2\eta - 108\partial\beta\beta\gamma\gamma\eta + \\
&\quad + 144\partial\beta\gamma\partial\eta - 144\partial\eta\eta\partial\xi - 36\partial^2\beta\gamma\eta + 36\partial^2\eta\eta\xi , \\
H^{--} &= 24\beta\beta\beta\gamma\gamma\gamma\gamma\eta - 144\beta\beta\gamma\gamma\gamma\partial\eta + 432\beta\gamma\gamma\partial\eta\eta\xi + 72\beta\gamma\gamma\partial^2\eta + 576\gamma\partial\eta\eta\partial\xi - \\
&\quad - 144\gamma\partial^2\eta\eta\xi + 144\partial\beta\beta\gamma\gamma\gamma\eta - 288\partial\beta\gamma\gamma\partial\eta + 72\partial^2\beta\gamma\gamma\eta .
\end{aligned} \tag{A.5}
$$

One can verify that:

$$
G^a(z)G^b(w) \sim 0 , \qquad H^a(z)H^b(w) \sim 0 \tag{A.6}
$$

so the only non-trivial OPEs are between G and H multiplets.

$$
\begin{aligned}
G^{++}(z)H^{++}(w) &\sim \frac{J^+J^+J^+J^+}{(z-w)^2} + \frac{2\partial J^+J^+J^+J^+}{(z-w)} , \\
G^{++}(z)H^{+}(w) &\sim \frac{12J^+J^+J^+}{(z-w)^3} + \frac{-4J^+J^+J^+J^3 + 12\partial J^+J^+J^+}{(z-w)^2} +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{z-w}\big(-4J^+J^+J^+J^3J^3-4J^+J^+J^+J^+J^--4J^+J^+J^+\partial J^3-\\
&\qquad -12\partial J^+\partial J^+J^++12\partial^2J^+J^+J^+\big)\,,\\
G^{++}(z)H^0(w)\sim\;&\frac{72J^+J^+}{(z-w)^4}+\frac{-72J^+J^+J^3}{(z-w)^3}+\\
&+\frac{-12J^+J^+J^+J^--72\partial J^+J^+J^3-72\partial J^+\partial J^++36\partial^2J^+J^+}{(z-w)^2}+\\
&+\frac{1}{z-w}\big(24J^+J^+J^3J^3J^3+24J^+J^+J^+J^3J^--12J^+J^+J^+\partial J^--\\
&\qquad -12J^+J^+\partial^2J^3+36\partial J^+J^+J^+J^-+72\partial J^+J^+\partial J^3-\\
&\qquad -72\partial^2J^+J^+J^3-36\partial^2J^+\partial J^++12\partial^3J^+J^+\big)\,,\\
G^{++}(z)H^-(w)\sim\;&\frac{144J^+}{(z-w)^5}+\frac{-432J^+J^3-144\partial J^+}{(z-w)^4}+\\
&+\frac{144J^+J^3J^3-72J^+J^+J^-+144J^+\partial J^3-288\partial J^+J^3}{(z-w)^3}+\\
&+\frac{1}{(z-w)^2}\big(48J^+J^3J^3J^3+72J^+J^+J^3J^-+24J^+\partial^2J^3+\\
&\qquad +144\partial J^+J^3J^3+288\partial J^+\partial J^3-216\partial^2J^+J^3\big)+\\
&+\frac{1}{z-w}\big(-54J^+J^3J^3J^3J^3-36J^+J^+J^3J^3J^-+108J^+J^+J^3\partial J^-+\\
&\qquad +18J^+J^+J^+J^-J^--72J^+J^+\partial J^3J^-+36J^+\partial J^3J^3J^3-\\
&\qquad -90J^+J^+\partial J^3\partial J^3+72J^+\partial^2J^3J^3-6J^+\partial^3J^3+\\
&\qquad +24\partial J^+J^3J^3J^3-108\partial J^+J^+J^3J^-+108\partial J^+J^+\partial J^--\\
&\qquad -72\partial J^+J^3J^3+24\partial J^+\partial^2J^3+108\partial^2J^+J^3J^3-\\
&\qquad -108\partial^2J^+J^+J^-+108\partial^2J^+\partial J^3-72\partial^3J^+J^3\big)\,,\\
G^{++}(z)H^{--}(w)\sim\;&\frac{-576J^3}{(z-w)^5}+\frac{288J^3J^3-576J^+J^-+288\partial J^3}{(z-w)^4}+\\
&+\frac{288J^+J^3J^-+288J^+\partial J^--576\partial J^+J^-}{(z-w)^3}+\\
&+\frac{1}{(z-w)^2}\big(-48J^3J^3J^3J^3+144J^+J^3\partial J^-+72J^+J^+J^-J^--\\
&\qquad -144J^+\partial J^3J^--144\partial J^3\partial J^3+144\partial J^+J^3J^-+\\
&\qquad +432\partial J^+\partial J^-+96\partial^2J^3J^3-432\partial^2J^+J^-\big)+\\
&+\frac{1}{z-w}\big(24J^3J^3J^3J^3J^3-48J^+J^3J^3J^3J^--288J^+J^3J^3\partial J^--\\
&\qquad -72J^+J^+J^3J^-J^-+288J^+\partial J^3J^3J^--144J^+\partial J^3J^-+\\
&\qquad +96J^+\partial^2J^3J^--48\partial J^3J^3J^3J^3+72\partial J^3\partial J^3J^3+\\
&\qquad +144\partial J^+J^3J^3J^--144\partial J^+\partial J^3J^--48\partial^2J^3J^3J^3-\\
&\qquad -48\partial^2J^3\partial J^3+216\partial^2J^+J^3J^-+216\partial^2J^+\partial J^-+\\
&\qquad +24\partial^3J^3J^3-144\partial^3J^+J^-\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
G^+(z)H^{++}(w) \sim\ & \frac{-12J^+J^+J^+}{(z-w)^3} + \frac{-4J^+J^+J^+J^3 - 24\partial J^+J^+J^+}{(z-w)^2} + \\
& + \frac{4J^+J^+J^+J^3J^3 + 4J^+J^+J^+J^+J^- - 12\partial J^+J^+J^+J^3 - 18\partial^2J^+J^+J^+}{(z-w)}, \\
G^+(z)H^+(w) \sim\ & \frac{-72J^+J^+}{(z-w)^4} + \frac{-72\partial J^+J^+}{(z-w)^3} + \\
& + \frac{1}{(z-w)^2}\big(24J^+J^+J^3J^3 + 8J^+J^+J^+J^- - 24J^+J^+\partial J^3 + \\
& \qquad + 48\partial J^+J^+J^3 + 48\partial J^+\partial J^+ - 60\partial^2J^+J^+\big) + \\
& + \frac{1}{z-w}\big(4J^+J^+J^+\partial J^- + 24J^+J^+\partial J^3J^3 - 12J^+J^+\partial^2J^3 + \\
& \qquad + 24\partial J^+J^+J^3J^3 + 12\partial J^+J^+J^+J^- + 24\partial J^+\partial J^+J^3 + \\
& \qquad + 24\partial^2J^+J^+J^3 + 36\partial^2J^+\partial J^+ - 24\partial^3J^+J^+\big), \\
G^+(z)H^0(w) \sim\ & \frac{-144J^+}{(z-w)^5} + \frac{144J^+J^3}{(z-w)^4} + \frac{144J^+J^3J^3 - 144J^+\partial J^3 + 432\partial J^+J^3}{(z-w)^3} + \\
& + \frac{-48J^+J^3J^3J^3 + 144J^+\partial J^3J^3 - 48J^+\partial^2J^3 + 216\partial^2J^+J^3}{(z-w)^2} + \\
& + \frac{1}{z-w}\big(-42J^+J^3J^3J^3J^3 - 36J^+J^+J^3J^3J^- + 36J^+J^+J^3\partial J^- + \\
& \qquad + 6J^+J^+J^+J^-J^- - 36J^+\partial J^3J^3J^3 - 54J^+\partial J^3\partial J^3 + \\
& \qquad + 120J^+\partial^2J^3J^3 - 18J^+\partial^3J^3 - 72\partial J^+J^3J^3J^3 - \\
& \qquad - 180\partial J^+J^+J^3J^- + 36\partial J^+J^+\partial J^- - 72\partial J^+\partial J^3J^3 - \\
& \qquad - 72\partial J^+\partial J^+J^- + 72\partial J^+\partial^2J^3 + 36\partial^2J^+J^3J^3 - \\
& \qquad - 36\partial^2J^+J^+J^- - 108\partial^2J^+\partial J^3 + 96\partial^3J^+J^3\big), \\
G^+(z)H^-(w) \sim\ & \frac{288J^3}{(z-w)^5} + \frac{576J^3J^3 + 144J^+J^-}{(z-w)^4} + \\
& + \frac{-288J^3J^3J^3 + 288J^+J^3J^- + 288\partial J^3J^3 + 432\partial J^+J^-}{(z-w)^3} + \\
& + \frac{1}{(z-w)^2}\big(-48J^3J^3J^3J^3 - 144J^+J^3J^3J^- + 144J^+\partial J^3J^- - \\
& \qquad - 288\partial J^3J^3J^3 - 432\partial J^3\partial J^3 + 384\partial^2J^3J^3 + \\
& \qquad + 216\partial^2J^+J^-\big) + \\
& + \frac{1}{z-w}\big(84J^3J^3J^3J^3J^3 - 24J^+J^3J^3J^3J^- - 432J^+J^3J^3\partial J^- - \\
& \qquad - 108J^+J^+J^3J^-J^- + 288J^+\partial J^3J^3J^- - 216J^+\partial J^3\partial J^- + \\
& \qquad + 192J^+\partial^2J^3J^- - 72\partial J^3J^3J^3J^3 + 252\partial J^3\partial J^3J^3 + \\
& \qquad + 216\partial J^+J^3J^3J^- - 432\partial J^+J^3\partial J^- - 216\partial J^+J^+J^-J^- + \\
& \qquad + 216\partial J^+\partial J^3J^- - 312\partial^2J^3J^3J^3 - 216\partial^2J^3\partial J^3 + \\
& \qquad + 324\partial^2J^+J^3J^- - 108\partial^2J^+\partial J^- + 132\partial^3J^3J^3 + \\
& \qquad + 144\partial^3J^+J^-\big),
\end{aligned}
$$

$$
\begin{aligned}
G^+(z)H^{--}(w) \sim\ & \frac{-576J^-}{(z-w)^5} + \frac{1728J^3J^- + 576\partial J^-}{(z-w)^4} + \\
& + \frac{-576J^3J^3J^- - 576J^3\partial J^- + 288J^+J^-J^- + 1152\partial J^3J^-}{(z-w)^3} + \\
& + \frac{1}{(z-w)^2}\big(-192J^3J^3J^3J^- - 576J^3J^3\partial J^- - 288J^+J^3J^-J^- - \\
& \qquad - 1152\partial J^3\partial J^- + 960\partial^2J^3J^-\big) + \\
& + \frac{1}{z-w}\big(216J^3J^3J^3J^3J^- + 768J^3J^3J^3\partial J^- + 144J^+J^3J^3J^-J^- - \\
& \qquad - 432J^+J^3\partial J^-J^- - 72J^+J^+J^-J^-J^- + 288J^+\partial J^3J^-J^- - \\
& \qquad - 432J^+\partial J^-\partial J^- + 216J^+\partial^2J^-J^- - 1008\partial J^3J^3J^3J^- + \\
& \qquad + 288\partial J^3J^3\partial J^- + 360\partial J^3\partial J^3J^- + 432\partial J^+J^3J^-J^- - \\
& \qquad - 720\partial^2J^3J^3J^- - 528\partial^2J^3\partial J^- + 216\partial^2J^+J^-J^- + \\
& \qquad +312\partial^3J^3J^-\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
G^0(z)H^{++}(w) \sim\ & \frac{72J^+J^+}{(z-w)^4} + \frac{72J^+J^+J^3 + 144\partial J^+J^+}{(z-w)^3} + \\
& + \frac{-12J^+J^+J^+J^- + 72J^+J^+\partial J^3 + 72\partial J^+J^+J^3 + 108\partial^2J^+J^+}{(z-w)^2} + \\
& + \frac{1}{z-w}\big(-24J^+J^+J^3J^3J^3 - 24J^+J^+J^+J^3J^- + 48J^+J^+\partial^2J^3 - \\
& \qquad - 72\partial J^+J^+J^+J^- + 72\partial^2J^+J^+J^3 + 48\partial^3J^+J^+\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
G^0(z)H^{+}(w) \sim\ & \frac{144J^+}{(z-w)^5} + \frac{144J^+J^3 + 144\partial J^+}{(z-w)^4} + \\
& + \frac{-144J^+J^3J^3 + 288J^+\partial J^3 - 288\partial J^+J^3 + 72\partial^2J^+}{(z-w)^3} + \\
& + \frac{1}{(z-w)^2}\big(-48J^+J^3J^3J^3 - 144J^+\partial J^3J^3 + 168J^+\partial^2J^3 - \\
& \qquad - 144\partial J^+J^3J^3 - 144\partial J^+\partial J^3 - 144\partial^2J^+J^3 + 24\partial^3J^+\big) + \\
& + \frac{1}{z-w}\big(42J^+J^3J^3J^3J^3 + 36J^+J^+J^3J^3J^- - 36J^+J^+J^3\partial J^- - \\
& \qquad - 6J^+J^+J^+J^-J^- - 108J^+\partial J^3J^3J^3 + 54J^+\partial J^3\partial J^3 - \\
& \qquad - 120J^+\partial^2J^3J^3 + 66J^+\partial^3J^3 + 24\partial J^+J^3J^3J^3 + \\
& \qquad + 180\partial J^+J^+J^3J^- - 36\partial J^+J^+\partial J^- - 72\partial J^+\partial J^3J^3 + \\
& \qquad + 72\partial J^+\partial J^+J^- - 120\partial J^+\partial^2J^3 - 108\partial^2J^+J^3J^3 + \\
& \qquad + 36\partial^2J^+J^+J^- + 36\partial^2J^+\partial J^3 - 72\partial^3J^+J^3 + 6\partial^4J^+\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
G^0(z)H^{0}(w) \sim\ & \frac{-864J^3J^3}{(z-w)^4} + \frac{-864\partial J^3J^3}{(z-w)^3} + \\
& + \frac{144J^3J^3J^3J^3 + 432\partial J^3\partial J^3 - 720\partial^2J^3J^3}{(z-w)^2} +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{288\partial J^3J^3J^3J^3+288\partial^2J^3\partial J^3-288\partial^3J^3J^3}{(z-w)}\,,\\
G^0(z)H^-(w)\sim{}&\frac{864J^-}{(z-w)^5}+\frac{-864J^3J^-}{(z-w)^4}+\frac{-864J^3J^3J^--1728\partial J^3J^-}{(z-w)^3}+\\
&+\frac{288J^3J^3J^3J^--864\partial J^3J^3J^--1008\partial^2J^3J^-}{(z-w)^2}+\\
&+\frac{1}{z-w}\big(252J^3J^3J^3J^3J^-+864J^3J^3J^3\partial J^-+216J^+J^3J^3J^-J^--\\
&\qquad-216J^+J^3\partial J^-J^--36J^+J^+J^-J^-J^-+432J^+\partial J^3J^-J^--\\
&\qquad-216J^+\partial J^-\partial J^-+108J^+\partial^2J^-J^--216\partial J^3J^3J^3J^-+\\
&\qquad+1296\partial J^3J^3\partial J^--540\partial J^3\partial J^3J^-+648\partial J^+J^3J^-J^--\\
&\qquad-936\partial^2J^3J^3J^-+216\partial^2J^3\partial J^-+324\partial^2J^+J^-J^--\\
&\qquad-468\partial^3J^3J^-\big)\,,\\
G^0(z)H^{--}(w)\sim{}&\frac{1728J^-J^-}{(z-w)^4}+\frac{-1728J^3J^-J^-}{(z-w)^3}+\\
&+\frac{-1728J^3\partial J^-J^--288J^+J^-J^-J^--1728\partial J^-\partial J^-+864\partial^2J^-J^-}{(z-w)^2}+\\
&+\frac{1}{z-w}\big(576J^3J^3J^3J^-J^-+3456J^3J^3\partial J^-J^-+3456J^3\partial J^-\partial J^--\\
&\qquad-1728J^3\partial^2J^-J^-+576J^+J^3J^-J^-J^--3456\partial J^3J^3J^-J^-+\\
&\qquad+576\partial J^+J^-J^-J^--1152\partial^2J^3J^-J^-\big)\,,\\
G^-(z)H^{++}(w)\sim{}&\frac{-144J^+}{(z-w)^5}+\frac{-432J^+J^3-288\partial J^+}{(z-w)^4}+\\
&+\frac{-144J^+J^3J^3+72J^+J^+J^--576J^+\partial J^3-144\partial J^+J^3-216\partial^2J^+}{(z-w)^3}+\\
&+\frac{1}{(z-w)^2}\big(48J^+J^3J^3J^3+72J^+J^+J^3J^-+72J^+J^+\partial J^--\\
&\qquad-288J^+\partial J^3J^3-384J^+\partial^2J^3+144\partial J^+J^+J^--\\
&\qquad-144\partial^2J^+J^3-96\partial^3J^+\big)+\\
&+\frac{1}{z-w}\big(54J^+J^3J^3J^3J^3+36J^+J^+J^3J^3J^--36J^+J^+J^3\partial J^--\\
&\qquad-18J^+J^+J^+J^-J^-+144J^+J^+\partial J^3J^-+36J^+J^+\partial^2J^-+\\
&\qquad+108J^+\partial J^3J^3J^3-54J^+\partial J^3\partial J^3-216J^+\partial^2J^3J^3-\\
&\qquad-162J^+\partial^3J^3+24\partial J^+J^3J^3J^3+252\partial J^+J^+J^3J^-+\\
&\qquad+36\partial J^+J^+\partial J^-+72\partial J^+\partial J^3J^3+72\partial J^+\partial J^+J^-+\\
&\qquad+24\partial J^+\partial^2J^3-36\partial^2J^+J^3J^3+180\partial^2J^+J^+J^--\\
&\qquad-36\partial^2J^+\partial J^3-72\partial^3J^+J^3-30\partial^4J^+\big)\,,\\
G^-(z)H^+(w)\sim{}&\frac{-288J^3}{(z-w)^5}+\frac{576J^3J^3+144J^+J^--288\partial J^3}{(z-w)^4}+
\end{aligned}
$$

$$+\frac{1}{(z-w)^3}\big(288J^3J^3J^3-288J^+J^3J^--+144J^+\partial J^-+$$

$$+864\partial J^3J^3-288\partial J^+J^--144\partial^2J^3\big)+$$

$$+\frac{1}{(z-w)^2}\big(-48J^3J^3J^3J^3-144J^+J^3J^3J^--288J^+J^3\partial J^--$$

$$-144J^+\partial J^3J^-+72J^+\partial^2J^-+576\partial J^3J^3J^3-$$

$$-144\partial J^3\partial J^3-288\partial J^+J^3J^--288\partial J^+\partial J^-+$$

$$+672\partial^2J^3J^3-144\partial^2J^+J^--48\partial^3J^3\big)+$$

$$+\frac{1}{z-w}\big(-84J^3J^3J^3J^3J^3+24J^+J^3J^3J^3J^-+288J^+J^3J^3\partial J^--$$

$$-144J^+J^3\partial^2J^-+108J^+J^+J^3J^-J^--576J^+\partial J^3J^3J^-+$$

$$+72J^+\partial J^3\partial J^--192J^+\partial^2J^3J^-+24J^+\partial^3J^--$$

$$-120\partial J^3J^3J^3J^3+36\partial J^3\partial J^3J^3-360\partial J^+J^3J^3J^-+$$

$$+144\partial J^+J^3\partial J^-+216\partial J^+J^+J^-J^--360\partial J^+\partial J^3J^--$$

$$-144\partial J^+\partial^2J^-+456\partial^2J^3J^3J^3-120\partial^2J^3\partial J^3-$$

$$-468\partial^2J^+J^3J^--36\partial^2J^+\partial J^-+300\partial^3J^3J^3-$$

$$-120\partial^3J^+J^--12\partial^4J^3\big)\,,$$

$$G^-(z)H^0(w)\sim\frac{-864J^-}{(z-w)^5}+\frac{-864J^3J^--864\partial J^-}{(z-w)^4}+$$

$$+\frac{864J^3J^3J^--864J^3\partial J^-+864\partial J^3J^--432\partial^2J^-}{(z-w)^3}+$$

$$+\frac{1}{(z-w)^2}\big(288J^3J^3J^3J^-+864J^3J^3\partial J^--432J^3\partial^2J^-+$$

$$+864\partial J^3J^3J^-+864\partial J^3J^3J^-+864\partial J^3\partial J^-+$$

$$+288\partial^2J^3J^--144\partial^3J^-\big)+$$

$$+\frac{1}{z-w}\big(-252J^3J^3J^3J^3J^--576J^3J^3J^3\partial J^-+432J^3J^3\partial^2J^--$$

$$-144J^3\partial^3J^--216J^+J^3J^3J^-J^-+216J^+J^3\partial J^-J^-+$$

$$+36J^+J^+J^-J^-J^--432J^+\partial J^3J^-J^-+216J^+\partial J^-\partial J^--$$

$$-108J^+\partial^2J^-J^-+1080\partial J^3J^3J^3J^--432\partial J^3J^3\partial J^-+$$

$$+540\partial J^3\partial J^3J^-+432\partial J^3\partial^2J^--648\partial J^+J^3J^-J^-+$$

$$+936\partial^2J^3J^3J^-+72\partial^2J^3\partial J^--324\partial^2J^+J^-J^-+$$

$$+180\partial^3J^3J^--36\partial^4J^-\big)\,,$$

$$G^-(z)H^-(w)\sim\frac{-2592J^-J^-}{(z-w)^4}+\frac{-2592\partial J^-J^-}{(z-w)^3}+$$

$$+\frac{1}{(z-w)^2}\big(864J^3J^3J^-J^-+1728J^3\partial J^-J^-+288J^+J^-J^-J^--$$

$$-864\partial J^3J^-J^-+1728\partial J^-\partial J^--2160\partial^2J^-J^-\big)+$$

$$+\frac{1}{z-w}\big(864J^3J^3\partial J^-J^-+864J^3\partial J^-\partial J^-+864J^3\partial^2J^-J^-+$$

$$+\ 432J^+\partial J^-J^-J^- + 864\partial J^3J^3J^-J^- + 144\partial J^+J^-J^-J^- -$$
$$-\ 432\partial^2J^3J^-J^- + 1296\partial^2J^-\partial J^- - 864\partial^3J^-J^-\big)\,,$$

$$G^-(z)H^{--}(w) \sim \frac{-1728J^-J^-J^-}{(z-w)^3} + \frac{576J^3J^-J^-J^- - 1728\partial J^-J^-J^-}{(z-w)^2} +$$
$$+\frac{1}{z-w}\big(576J^3J^3J^-J^-J^- + 5184J^3\partial J^-J^-J^- + 576J^+J^-J^-J^-J^- -$$
$$-\ 4608\partial J^3J^-J^-J^- + 6912\partial J^-\partial J^-J^- - 4320\partial^2J^-J^-J^-\big)\,,$$

$$G^{--}(z)H^{++}(w) \sim \frac{576J^3}{(z-w)^5} + \frac{288J^3J^3 - 576J^+J^- + 864\partial J^3}{(z-w)^4} +$$
$$+\ \frac{-288J^+J^3J^- - 864J^+\partial J^- + 576\partial J^3J^3 + 576\partial^2J^3}{(z-w)^3} +$$
$$+\ \frac{1}{(z-w)^2}\big(-48J^3J^3J^3J^3 - 144J^+J^3\partial J^- + 72J^+J^+J^-J^- -$$
$$-\ 432J^+\partial J^3J^- - 576J^+\partial^2J^- + 144\partial J^3\partial J^3 -$$
$$-\ 144\partial J^+J^3J^- + 144\partial J^+\partial J^- + 384\partial^2J^3J^3 -$$
$$-\ 144\partial^2J^+J^- + 240\partial^3J^3\big) +$$
$$+\ \frac{1}{z-w}\big(-24J^3J^3J^3J^3J^3 + 48J^+J^3J^3J^3J^- + 288J^+J^3J^3\partial J^- +$$
$$+\ 72J^+J^+J^3J^-J^- + 144J^+J^+\partial J^-J^- - 288J^+\partial J^3J^3J^- -$$
$$-\ 144J^+\partial J^3\partial J^- - 384J^+\partial^2J^3J^- - 240J^+\partial^3J^- -$$
$$-\ 144\partial J^3J^3J^3J^3 - 72\partial J^3\partial J^3J^3 - 144\partial J^+J^3J^3J^- +$$
$$+\ 144\partial J^+J^+J^-J^- - 144\partial J^+\partial J^3J^- + 144\partial J^+\partial^2J^- +$$
$$+\ 48\partial^2J^3J^3J^3 + 144\partial^2J^3\partial J^3 - 216\partial^2J^+J^3J^- -$$
$$-\ 72\partial^2J^+\partial J^- + 168\partial^3J^3J^3 - 96\partial^3J^+J^- + 72\partial^4J^3\big)\,,$$

$$G^{--}(z)H^{+}(w) \sim \frac{576J^-}{(z-w)^5} + \frac{1728J^3J^- + 1152\partial J^-}{(z-w)^4} +$$
$$+\ \frac{1}{(z-w)^3}\big(576J^3J^3J^- + 2304J^3\partial J^- - 288J^+J^-J^- +$$
$$+\ 576\partial J^3J^- + 864\partial^2J^-\big) +$$
$$+\ \frac{1}{(z-w)^2}\big(-192J^3J^3J^3J^- + 1440J^3\partial^2J^- - 288J^+J^3J^-J^- -$$
$$-\ 576J^+\partial J^-J^- + 1152\partial J^3J^3J^- - 288\partial J^+J^-J^- +$$
$$+\ 672\partial^2J^3J^- + 384\partial^3J^-\big) +$$
$$+\ \frac{1}{z-w}\big(-216J^3J^3J^3J^3J^- - 960J^3J^3J^3\partial J^- - 288J^3J^3\partial^2J^- +$$
$$+\ 576J^3\partial^3J^- - 144J^+J^3J^3J^-J^- - 144J^+J^3\partial J^-J^- +$$
$$+\ 72J^+J^+J^-J^-J^- - 576J^+\partial J^3J^-J^- + 144J^+\partial J^-\partial J^- -$$
$$-\ 504J^+\partial^2J^-J^- + 432\partial J^3J^3J^3J^- - 288\partial J^3J^3\partial J^- +$$
$$+\ 216\partial J^3\partial J^3J^- - 288\partial J^3\partial^2J^- - 720\partial J^+J^3J^-J^- -$$

$$
\begin{aligned}
&\quad - 576\partial J^+ \partial J^- J^- + 1296\partial^2 J^3 J^3 J^- + 336\partial^2 J^3 \partial J^- - \\
&\quad - 360\partial^2 J^+ J^- J^- + 360\partial^3 J^3 J^- + 120\partial^4 J^- \big)\,, \\
G^{--}(z)H^0(w) \sim{}& \frac{1728 J^- J^-}{(z-w)^4} + \frac{1728 J^3 J^- J^- + 3456\partial J^- J^-}{(z-w)^3} + \\
&+ \frac{1}{(z-w)^2}\big(1728 J^3 \partial J^- J^- - 288 J^+ J^- J^- J^- + \\
&\quad + 1728\partial J^3 J^- J^- + 2592\partial^2 J^- J^-\big) + \\
&+ \frac{1}{z-w}\big(-576 J^3 J^3 J^3 J^- J^- - 3456 J^3 J^3 \partial J^- J^- - \\
&\quad - 3456 J^3 \partial J^- \partial J^- + 1728 J^3 \partial^2 J^- J^- - 576 J^+ J^3 J^- J^- J^- - \\
&\quad - 864 J^+ \partial J^- J^- J^- + 3456\partial J^3 J^3 J^- J^- + 1728\partial J^3 \partial J^- J^- - \\
&\quad - 864\partial J^+ J^- J^- J^- + 2016\partial^2 J^3 J^- J^- - 864\partial^2 J^- \partial J^- + \\
&\quad + 1440\partial^3 J^- J^-\big)\,, \\
G^{--}(z)H^-(w) \sim{}& \frac{1728 J^- J^- J^-}{(z-w)^3} + \frac{576 J^3 J^- J^- J^- + 3456\partial J^- J^- J^-}{(z-w)^2} + \\
&+ \frac{1}{z-w}\big(-576 J^3 J^3 J^- J^- J^- - 3456 J^3 \partial J^- J^- J^- - \\
&\quad - 576 J^+ J^- J^- J^- J^- + 5184\partial J^3 J^- J^- J^- - \\
&\quad - 5184\partial J^- \partial J^- J^- + 5184\partial^2 J^- J^- J^-\big)\,, \\
G^{--}(z)H^{--}(w) \sim{}& \frac{576 J^- J^- J^- J^-}{(z-w)^2} + \frac{1152\partial J^- J^- J^- J^-}{(z-w)}\,.
\end{aligned}
\tag{A.7}
$$

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